Proceedings of IAM, V.11, N.1, 2022, pp.45-49

SUFFICIENT CONDITIONS FOR EXISTENCE OF PERIODIC SOLUTION OF A NON-LINEAR DYNAMICAL SYSTEM OF SECOND ORDER

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Abstract. In this paper we consider anon-linear dynamical system as aB.V.P that includes asystem of second order differential equations with dimensions nwith periodic boundary conditions and we willshow that the definedintegral operator has at least one fixed point in asuitable functional space.

Keywords: Green-function, Periodic conditions, Fixed point theorems, nonlinear, dynamic system, second order

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AMS Subject Classification: 13D45, 39B42

Introduction

Due to the existence of cyclical natural phenomena in the world around us, such as the cycle of night and day, the cycle of planetary motion around the sun, the cycle of seasons and even the cycle of life itself (birth and death), it is necessary to study these phenomena. Therefore, studying mathematical models of these phenomena and studying periodic solutions and analyzing and interpreting the answers is of great importance [8].

In the theory of boundary and initial value problems including ordinary and partial differential equations, integral equations have important role in solving of this problems [3]. Recently, the fixed point theory has successfully been used to deal with the existence of positive solutions to boundary value problems. In this problems usually B.V.P and I.V.P are been reduced to Fredholm and Voltera Integral equations respectively [2]. In this paper we consider anon-linear dynamical system as aB.V.P that includes asystem of second order differential equations with dimensions nwith periodic boundary conditions.At firs for linear part of equations we form G reen-function of homogeneous linear problem. Then we define a Fredholm integral operator with its kernel is above mentioned Green-function. Finally we show that the mentioned operator has at least one fixed point in asuitable functional space[6], [7].

These problems are appear in many physical and engineering systems such as motion of ship and fluid dynamics considerations.For example, the following sy stem describes the three-dimensional of the Laser plume,

$$x(t)\frac{d^2x(t)}{dt^2}=y(t)\frac{d^2y(t)}{dt^2}=z(t)\frac{d^2z(t)}{dt^2}=\frac{KT_0}{m}[\frac{x_0y_0z_0}{x(t)y(t)z(t)}]^{\gamma-1} \qquad (1)$$

Where $x(t), y(t)$and $z(t)$are the dependent dimensions of the plasma in the adiabatic regime and $x_0, y_0$and z_0 corresponde to the orthogonal dimensions of the plasma at the end of the isothermal regime and $K, m$and T_0 are others physical constants [9] .

1. Mathematical statement of problem

We consider the following dynamical system:

$$\ddot{x}(t)+Ax(t)=f\big(t,x(t)\big), \quad t\in(0,\omega) \qquad (2)$$

$$x(0)=x(\omega), \dot{x}(0)=\dot{x}(\omega) \qquad (3)$$

where $x(t)$is column vector and \$$A$is a $n\times\ n$matrix as follows,

$$x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\\\vdots\\x_n(t)\end{bmatrix}, A=\begin{bmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{n1}&\cdots&a_{nn}\end{bmatrix}, f\big(t,x(t)\big)=\begin{bmatrix}f_1(t,x(t))\\f_2(t,x(t))\\\vdots\\f_n(t,x(t))\end{bmatrix}$$

Moreover, $A$is symmetric matrix and its elements are real constants.
Firstly, we consider polynomial characteristice equation of A, that is,

$$|A-\lambda I|=0 \qquad (4)$$

We assume that the roots of (4) are distinct and positive, that is,

$$\lambda_k>0\ ,k=1,2,\dots,n\ ,\lambda_i\neq\lambda_j\ ,\text{if } i\neq j \qquad (5)$$

Under these conditions, there is a nonsingular matrix Psuch that,$\det P\neq 0$ and .
Matrix Bhas the diagonal form as follows,

$$B=\begin{bmatrix}\lambda_1&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&\lambda_n\end{bmatrix}$$

We let $x(t)=Py(t)$. Then we obtain form (2),

$$P\ddot{y}(t)+APy(t)=f\big(t,Py(t)\big)$$

If the two sides of this equation is multiplied byP^{-1}we have,

$$\ddot{y}(t)+By(t)=g\big(t,y(t)\big)$$

Where $g\big(t,y(t)\big)=P^{-1}f\big(t,Py(t)\big)$. Since $\lambda_k$are positive numbers, we can write $\lambda_k=b_k^2\,,b_k>0\,,k=1,2,\dots,n,$

2. Methods of Green function .

Then the system (2) can be written as follows,

$$\ddot{y}_j(t)+b_j^2y_j(t)=g_j\big(t,y(t)\big),\ \ j=1,2,\dots,n \tag{6}$$

$$y(0)=y(\omega),\dot{y}(0)=\dot{y}(\omega) \tag{7}$$

Now we are going to form the Green-function of problem (6). To do this, we want to find a particular solutionfor non-homogenous equation (6) by Lagrange method. Let,

$$y_j(t)=C_{1j}\cos b_j\,t+C_{2j}\sin b_j\,t,\quad j=1,2,\dots,n \tag{8}$$

So we have,

$$C_{1j}=\int_0^{\omega}\frac{\cos b_j\,(\frac{\omega}{2}-z)}{2W_j\sin b_j\frac{\omega}{2}}\,g_j\big(z,y(z)\big)dz$$

$$C_{2j}=\int_0^{\omega}\frac{\sin b_j\,(\frac{\omega}{2}-z)}{2W_j\sin b_j\frac{\omega}{2}}\,g_j\big(z,y(z)\big)dz$$

Where $W_j$is Wroneskian determinant and it is easily to see that $W_j=b_j$for $j=1,2,\dots,n$. By substitutingof this in (2.7) and some calculations we have,

$$y_j(t)=\int_0^{\omega}\frac{\cos b_j\,(|t-z|\frac{\omega}{2})}{2W_j\sin b_j\frac{\omega}{2}}\,g_j\big(z,y(z)\big)dz \tag{9}$$

Now from (9), we conclude the Green-function of (6) is,

$$G_j(t,z)=\frac{\cos b_j\,(|t-z|-\frac{\omega}{2})}{2W_j\sin b_j\frac{\omega}{2}}\ ,\qquad j=1,2,\dots,n \tag{10}$$

Suppose that the constant ωbe such that we would have,

$$0<b_j\omega<\pi,\qquad j=1,2,\dots,n \tag{11}$$

Then from (2.8) and the Green-function (2.5) we have,

$$g_j(t,\tau)>0,\qquad j=1,2,\dots,n$$

We suppose the function $g_j\big(t,y(t)\big)$ is a continuous and positive function, therefore this function gets its extremums on any compact set. Hence if we let,

$K=\{y(t)=\big(y_1(t),y_2(t),\dots,y_n(t)\big):0\le y_j(t)\le M,j=1,2,\dots,n\}$

Then,

$$\forall\big(t, y(t)\big) \in [0, \omega] \Longrightarrow 0 < g_j(t, y(t)) < C \tag{12}$$

Theorem 1. *Let* $(B, \|.\|)$*be a Banach space over* K*that* $(K = \mathbb{R}\text{or}\mathbb{C})$*and* $S \subseteq B$ *is closed, bounded, convex, and nonempty. Any compact operator* $T : S \longrightarrow S$ *has at least one fixed point.*
Proof.See [8] .
Now, we define the Fredholm integral operator on a Banach space Bwhich defined as follows,

$$\big(G_j y\big)(t) = \int_0^{\omega} G_j(t, \tau) g_j\big(\tau, y(\tau)\big) d\tau \tag{13}$$

$$B = \{y(t) \colon y \in C^2[0, \omega], y(0) = y(\omega), \dot{y}(0) = \dot{y}(\omega)\} \tag{14}$$

with this norm,

$$\|y\| = \sum_{j=1}^{n} \max\left[\left|y_j(t)\right| + \left|\dot{y}_j(t)\right| + \left|\ddot{y}_j(t)\right|\right], \qquad t \in [0, \omega] \tag{15}$$

We consider the functional space $B$is a Banach space and the operator $G_i \colon B \longrightarrow B$is a compact operator.We show that the Banach space B with operator $\big(G_j y\big)_{(i)}$satisfies the conditions of Schauder fixed point theorem (3). One can show that:

$$1) \int_0^{\omega} G_j(t, \tau) d\tau = \frac{1}{b_j^2} = constant \tag{16}$$

$$2) \big(G_j y\big)_{(i)} = \frac{C}{b_j^2} < M \tag{17}$$

Then we will have $0 < G_j y \leq M$. It means that the operator $G_j$operation K. To sum up we conclude thefollowing theorems.
Theorem 2. *Under hypothesis (17), (12) and (11) the operator* $\big(G_j y\big)(t)$*has at least one fixed point in*B.

Since by other hand the Fredholm integral equation (2.16) is equivalent to the problem (6)-(7), thereforewe conclude the following theorem:
Theorem 3. *Suppose in the problem (6)-(77), the constants* ω, C *and* M *would be as following:*

$$0 < b_j \omega < \pi \ \text{ and } \ \frac{C}{b_j^2} < M, \quad j = 1,2, \dots, n \tag{18}$$

And the function $g_j(t, y(t))$ are continuous and positive functions. Then the problem(6)-(7) has at least one periodic solution with periodω.

3. Conclusion

Now we would like to come back the main problem (2)-(3). For this we remember the changing variable$x(t) = Py(t)$. Finally by considering of above mentioned explains, existence of a periodic solution of problem(2)-(3) is proved. As mentioned in abstract we can consider the system (1) which it is appeared in many fields of physics and engineering. The dynamical system (1) is an Autonomous system. The existence ofperiodic solution of this system can be investigated by a similar manner.

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